MATH 2050 C   Lecture 1   (Jan 10)

Pre-requisites: MATH 1050/1058 (and MATH 1010/1018)

- Set theoretic concepts $(\forall, \exists, \in, \subseteq, \cap, \cup)$
- Number systems $(\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R})$
- Functions $f: A \to B$ [Ref: Chap. 1]

*. Proof writing

__Thm__: $\nexists\, r \in \mathbb{Q}$ s.t. $r^2 = 2$.   [i.e. $\sqrt{2}$ is irrational.]

__Proof__: We will prove "by contradiction".

Suppose __NOT__. Then, $\exists\, r \in \mathbb{Q}$ s.t. $r^2 = 2$.

Since $r \in \mathbb{Q}$, we can find $p, q \in \mathbb{Z}$, $q \neq 0$ s.t.

$$r = \frac{p}{q} \qquad \text{where } p, q \text{ are "relatively prime".}$$

- As $2 = r^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2} \implies \boxed{p^2 = 2q^2} \qquad (\#)$

  i.e. p^2 is even $\implies$ p is even, i.e. $\exists\, k \in \mathbb{Z}$ s.t. $p = 2k$.

- Plug $p = 2k$ into $(\#)$.

$$4k^2 = p^2 = 2q^2 \implies \boxed{q^2 = 2k^2} \qquad (\#\#)$$

Similar argument $\implies q^2$ is even $\implies q$ is even

Thus, both p & q are even, which __contradicts__ the fact that they are relatively prime.
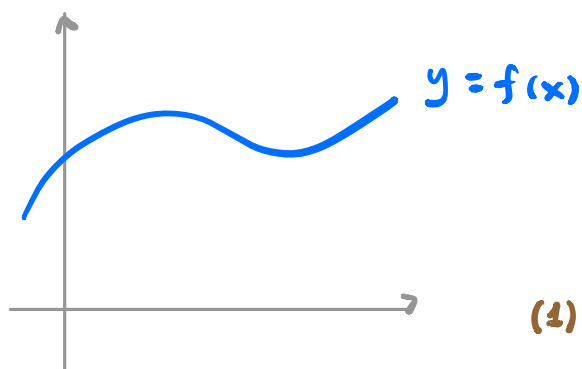
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An Overview of MATH 2050 (and 2060/3060)

Goal: Study the "analytic properties" of functions $f : \mathbb{R} \to \mathbb{R}$

[e.g. limit, continuity, differentiable / integrable ?]

$\underbrace{\qquad}_{\text{MATH 2050}}$ $\underbrace{\qquad}_{\text{MATH 2060 (3060)}}$

$y = f(x)$

Q: $\exists\ f : \mathbb{R} \to \mathbb{R}$ s.t. continuous everywhere but nowhere differentiable?

Summary (MATH 2050)

(1) [Ch. 2] $\mathbb{R}$ as complete ordered field.

(2) [Ch. 3] limit of sequences $\lim (x_n)$

(3) [Ch. 4] limit of functions $\lim_{x \to a} f(x)$

(4) [Ch. 5] continuity of functions

Chapter 2   The Real Numbers

Grand Thm: $\mathbb{R}$ is a complete ordered field.

— analysis (topology)

— inequalities

— algebra

Field Properties

Def^n/ Thm : $(\mathbb{R}, +, \cdot)$ is a field, i.e.

$\exists$ two operations $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ s.t.

the following properties hold:

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(A1) $a + b = b + a$ $\forall a, b \in \mathbb{R}$

(A2) $(a + b) + c = a + (b + c)$ $\forall a, b, c \in \mathbb{R}$

(A3) $\exists \; 0 \in \mathbb{R}$ s.t. $0 + a = a = a + 0$ $\forall a \in \mathbb{R}$

(A4) $\forall a \in \mathbb{R}, \exists -a \in \mathbb{R}$ s.t. $a + (-a) = 0 = (-a) + a$.

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$\cdot$ {

(M1) $a \cdot b = b \cdot a$ $\forall a, b \in \mathbb{R}$

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ $\forall a, b, c \in \mathbb{R}$

(M3) $\exists \; 1 \in \mathbb{R}$ s.t. $1 \neq 0$ and $1 \cdot a = a = a \cdot 1$ $\forall a \in \mathbb{R}$

(M4) $\forall a \in \mathbb{R}, a \neq 0, \exists \frac{1}{a} \in \mathbb{R}$ s.t. $\frac{1}{a} \cdot a = 1 = a \cdot \frac{1}{a}$ $\forall a \in \mathbb{R}$

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$+$
$\cdot$ {

(D) $a \cdot (b + c) = a \cdot b + a \cdot c$ $\forall a, b, c \in \mathbb{R}$

$(b + c) \cdot a = b \cdot a + c \cdot a$ $\forall a, b, c \in \mathbb{R}$

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Note: The remaining algebraic properties can be deduced from the field properties above.

Define: $a - b := a + (-b)$. $\quad \frac{a}{b} := a \cdot \left(\frac{1}{b}\right)$

Notation: $a^n := \underbrace{a \cdot a \cdots a}_{n \text{ times}}$; $a^0 := 1$; $a^{-1} := \frac{1}{a}$
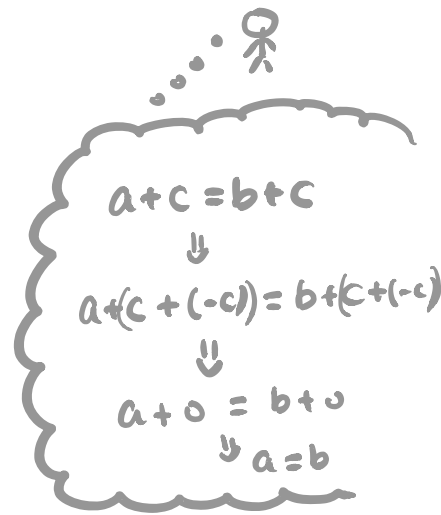
$n \in \mathbb{N}$

Prop: "Cancellation Laws"

(1) $\quad a + c = b + c \implies a = b$

(2) $\quad ac = bc$, $\boxed{c \neq 0} \implies a = b$

Proof: (1)

$$a = a + 0 \qquad \text{(by (A3))}$$
$$= a + (c + (-c)) \qquad \text{(by (A4))}$$
$$= (a + c) + (-c) \qquad \text{(by (A2))}$$
$$= (b + c) + (-c) \qquad \text{(by assumption)}$$
$$= b + (c + (-c)) \qquad \text{(by (A2))}$$
$$= b + 0 \qquad \text{(by (A4))}$$
$$= b \qquad \text{(by (A3))}$$

$a + c = b + c$
$\Downarrow$
$a + (c + (-c)) = b + (c + (-c))$
$\Downarrow$
$a + 0 = b + 0$
$\Downarrow a = b$

(2) : Exercise.

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Cor: The zero element 0 in (A3) is unique.

Proof: Suppose there are two zero elements 0, $0'$. Then

$$0 = 0 + 0' = 0' \qquad \text{i.e. } 0 = 0'$$

$0'$ (A3) $\qquad 0$ (A3)

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Exercise: 1 in (M3) is <u>unique</u>.

Prop: (1) $0 \cdot a = 0 \quad \forall a \in \mathbb{R}$

(2) $a \cdot b = 0 \Rightarrow a = 0 \text{ or } b = 0$ (or both)

(3) $(-1) \cdot a = -a \quad \forall a \in \mathbb{R}$

Proof: (1) Consider

$$0 \cdot a + 0 \cdot a \overset{(D)}{=} (0+0) \cdot a \overset{(A3)}{=} 0 \cdot a \overset{(A3)}{=} 0 \cdot a + 0$$

then by cancellation law (1), we have $0 \cdot a = 0$.

(2) Suppose $a \cdot b = 0$.

  <u>Case i</u> : $a = 0 \qquad \Rightarrow$ Done.

  <u>Case ii</u> : $a \neq 0 \qquad$ [Want: Prove $b = 0$.]

  Since $\underline{a \neq 0}$, the inverse $\frac{1}{a} \in \mathbb{R}$ exists.

$$a \cdot b = 0 = a \cdot 0$$

  by assumption $\qquad$ by (1)

  By cancellation law (2), we have $b = 0$.

(3) <u>Want to show:</u> $a + (-1) \cdot a = 0$

  Then, result follows from uniqueness of additive inverse "$-a$".

$$a + (-1) \cdot a \overset{(M3)}{=} 1 \cdot a + (-1) \cdot a$$
$$\overset{(D)}{=} (1 + (-1)) \cdot a$$
$$\overset{(A4)}{=} 0 \cdot a$$
$$\overset{by(1)}{=} 0$$

Remark: Other e.g. of fields $\mathbb{Q}, \mathbb{C}, \mathbb{Z}_p, \left\{ \frac{\text{polynomials}}{\text{polynomials}} \right\}$ .....